Central schemes for hydrodynamical limits of discrete-velocity kinetic models

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Abstract

The hydrodynamical scalings of many discrete-velocity kinetic models leads to a small-relaxation time behavior governed by the corresponding Euler type hyperbolic equations or Navier-Stokes type parabolic equations. Using as a prototype a simple discrete-velocity model of the Boltzmann equation we develop a class of central schemes with the correct asymptotic limit that work with uniform second order accuracy with respect to the scaling parameter. Numerical results for both the fluid-dynamic limit and the diffusive limit show the robustness of the present approach.

1 Introduction

Many discrete-velocity models of the Boltzmann equation have hydrodynamical scalings that leads to the Euler type hyperbolic equations or the Navier-Stokes type parabolic equations. These include scalar conservation laws, the heat equation, the advection-diffusion equation and the Burgers equation. In such problems the scaling parameter may differ in several orders of magnitude from the rarefied regimes to the hydrodynamic regimes, and it is desirable to develop a class of numerical schemes that can work uniformly with respect to this relaxation parameter.

In order for a scheme to be useful not only in the rarefied regime (where the relaxation parameter is $O(1)$) but also in the hydrodynamic regime (where the relaxation parameter is small) allowing the use of under-resolved space and time discretizations, the study of the numerical hydrodynamical limits has a critical importance.

In fact, the scheme should possess the correct hydrodynamical limit, in the sense that the that the asymptotic limit that leads from the kinetic equations to the hydrodynamic (Euler, Navier-Stokes or diffusion) equations should be preserved at the discrete level.

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Past progress on developing robust numerical schemes for kinetic equations has been strongly guided by the study of the hydrodynamical limits [1, 3, 6, 7, 8, 9, 11, 12].

In this paper we show how to develop a class of numerical schemes for some discrete-velocity kinetic models that work uniformly with respect to the relaxation parameter. While earlier approaches (cfr. [1, 6, 7, 8, 12]) that concentrated on getting the correct asymptotic limits were based on a Runge-Kutta technique to get uniform second order accuracy in time, here our key idea is to use a predictor-corrector approach to achieve the same goal.

Although the present approach does not necessarily require the use of the predictor-corrector central schemes for hyperbolic conservation laws recently developed by Nessyahu and Tadmor, these schemes will provide a natural starting point for the construction of our methods.

The rest of the paper is organized as follows. The next section is devoted to a description of the kinetic models under consideration and the variety of limiting equations we can obtain. In section 3 we give the details of our numerical schemes applied to the Euler scaling. Non oscillatory central schemes for the convection terms are combined with an implicit integration of the stiff source term based on a linear interpolation which depends on the relaxation parameter. The extension of the schemes to the Navier-Stokes (diffusive) scaling is discussed in section 4. In particular this requires the introduction of a modified Nessyahu-Tadmor scheme for hyperbolic conservation laws. Numerical results for both the Euler scaling and the diffusive scaling are presented in section 5. Some final considerations are reported in section 6.

2 Hydrodynamical limits of discrete-velocity kinetic models

The simplest one-dimensional discrete-velocity models of the Boltzmann equation are certainly those with two velocities. These models describe the evolution of the velocity distribution of a fictitious gas composed of two kinds of particles that move parallel to the x-axis with constant speed equal to one, either in the positive x-direction with a density u, or in the negative x-direction with a density v. The corresponding dimensionless hyperbolic system is given by the Ruijgrok-Wu model [15]

$$\begin{aligned} \nu\partial_t u + \partial_x u &= \frac{1}{\epsilon} J_\nu(u,v), && (1)\\ \nu\partial_t v - \partial_x v &= -\frac{1}{\epsilon} J_\nu(u,v), && (2) \end{aligned}$$

where $\nu > 0$ is the Mach number, $\epsilon > 0$ is the Knudsen number and the collision operator J has the form

$$J(u,v) = \{-\alpha_\nu u + \beta_\nu v + \gamma_\nu uv\}, \qquad (3)$$

with α_ν, β_ν and γ_ν nonnegative real constants.

From (3) three collision rules are allowed, which can be described by $+- \to ++$ with a rate γ_ν, $+ \to -$ with a rate α_ν, and $- \to +$ with a rate β_ν. When $\gamma_\nu > 0$, the presence of the process $+- \to ++$, together with the absence of $++ \to +-$, means the violation of the detailed balance, and time reversal invariance. This phenomenon obviously reflects also in the limit equation. We remark that uniqueness of the equilibrium distribution is guaranteed only if $\alpha_\nu \leq \beta_\nu$ [4, 15].

The macroscopic variables of the model are the local mass density ρ and the flux j defined by

$$\rho = u + v, \quad j = (u - v)/\nu.$$

Adding and subtracting the two equations in (1-2) one obtains the following equivalent system of macroscopic equations

$$\begin{aligned} \partial_t \rho + \partial_x j &= 0, & (4)\\ \partial_t j + \frac{1}{\nu^2}\partial_x \rho &= \frac{1}{\nu\epsilon}\hat{J}_\nu(\rho, j), & (5) \end{aligned}$$

where

$$\hat{J}_\nu(\rho, j) = \left\{ \rho(\beta - \alpha) - j(\alpha + \beta) + \frac{\gamma}{2}(\rho^2 - \nu^2 j^2) \right\}, \tag{6}$$

and the rescaled collision rates for $0 < \nu \leq 1$ are

$$\alpha_\nu = [\alpha(1 + \nu) + \beta(1 - \nu)]/2, \quad \beta_\nu = [\alpha(1 - \nu) + \beta(1 + \nu)]/2, \quad \gamma_\nu = \nu\gamma. \tag{7}$$

As for the Boltzmann equation there are two classical and natural asymptotic regimes [2]. The first one is obtained for fixed ν, namely

$$\nu = 1, \qquad \epsilon \to 0, \tag{8}$$

and leads to the corresponding compressible Euler equations (known as the *fluid-dynamic limit*). The second one is obtained for ν of the order of ϵ, that is

$$\nu = \epsilon \to 0, \tag{9}$$

which leads to the corresponding incompressible Navier-Stokes equations (known as the *diffusive limit*).

The scaling (8) in (5) leads to $\hat{J}_1(\rho, j) = 0$ which defines the local equilibrium state

$$j = F(\rho),$$

where $F(\rho)$ is given by [4]

$$F(\rho) = \begin{cases} \dfrac{1}{\gamma}\left\{ \left[(\gamma\rho)^2 + 2\gamma(\beta - \alpha)\rho + (\beta + \alpha)^2\right]^{\frac{1}{2}} - (\beta + \alpha) \right\} & \gamma > 0, \\[2ex] \dfrac{\beta - \alpha}{\alpha + \beta}\rho & \gamma = 0. \end{cases} \tag{10}$$

Thus, equation (4) relaxes towards the hyperbolic conservation law

$$\partial_t \rho + \partial_x F(\rho) = 0\,. \tag{11}$$

The second scaling, which corresponds to $\nu = \epsilon$ in (5), leads to $\hat{J}_0(\rho, j) = \partial_x \rho$ which defines the local equilibrium

$$j = F(\rho, \partial_x \rho),$$

where

$$F(\rho, \partial_x \rho) = \frac{1}{\alpha + \beta}\left[\rho(\beta - \alpha) + \frac{\gamma}{2}\rho^2 - \partial_x \rho\right]. \tag{12}$$

In this case equation (4) gives the second-order parabolic equation

$$\partial_t \rho + \partial_x F(\rho, \partial_x \rho) = 0\,. \tag{13}$$

It is remarkable that, even for this simple kinetic model, the variety of equations of continuum theory we can obtain in the limit is very rich. The arguments we have given are clearly formal, for a rigorous derivation of these asymptotic limits for two-velocity models of the Boltzmann equation we refer to [4, 10] and the references therein.

3 Central schemes for the fluid-dynamic limit

In this section we discuss the discretization of the kinetic systems introduced in section 2. As already observed, the main idea is to use a predictor-corrector approach for the construction of accurate and stable discretizations that work with a uniform accuracy for all range of the relaxation time. To this aim it is natural to use as a building block of the schemes the family of predictor-corrector, second order, central difference approximations recently introduced by Nessyahu and Tadmor [13].

Let the spatial grid points be x_i, for $i = \ldots, -1, 0, 1, \ldots$, with uniform mesh spacing $\Delta x = x_{i+1} - x_i$. The discrete time levels t^n are also spaced uniformly with $\Delta t = t^{n+1} - t^n$ for $n = 0, 1, 2, \ldots$. We denote by $w_i^n = w(x_i, t^n)$ the nodal values of the function w.

If we consider a one-dimensional system of conservation laws

$$\partial_t w + \partial_x (f(w)) = 0, \tag{14}$$

where w and f are N-dimensional vectors, the class of predictor-corrector central schemes developed in [13] is based on a staggered form of the Lax-Friedrichs scheme and reads

$$w_i^{n+\frac{1}{2}} = w_i^n - \frac{\lambda}{2} f_i', \tag{15}$$

$$w_{i+\frac{1}{2}}^{n+1} = \frac{1}{2}[w_i^n + w_{i+1}^n] + \frac{1}{8}[w_i' - w_{i+1}'] - \lambda[f(w_{i+1}^{n+\frac{1}{2}}) - f(w_i^{n+\frac{1}{2}})], \tag{16}$$

where $\lambda = \Delta t/\Delta x$ and f'_i and w'_i are numerical derivatives that satisfies the accuracy conditions

$$\frac{1}{\Delta x} f'_i = \partial_x f(w_i^n) + O(\Delta x), \quad \frac{1}{\Delta x} w'_i = \partial_x w_i^n + O(\Delta x). \tag{17}$$

The main advantage of these schemes is their simplicity and generality: no Riemann problems are solved and hence field-by-field decompositions are avoided.

Now we will apply the same technique to derive central schemes for a system like (4-5) under the scaling (8). This system can be written in compact form as

$$\partial_t w + \partial_x (f(w)) = \frac{1}{\epsilon} s(w), \tag{18}$$

where w, f and s are two dimensional vectors given by $w = (\rho, j)$, $f(w) = (j, \rho)$ and $s(w) = (0, \hat{J}_1)$.

In the predictor step the backward Euler method applied to the stiff source term suffices to give the desired accuracy and stability property. Thus our predictor step has the form

$$w_i^{n+\frac{1}{2}} = w_i^n - \frac{\lambda}{2} f'_i + \frac{\Delta t}{2\epsilon} s(w_i^{n+\frac{1}{2}}). \tag{19}$$

Note that this step will project the solution over the correct asymptotic limit since as $\epsilon \to 0$ we get the equation $s(w_i^{n+1/2}) = 0$ which gives the local equilibrium

$$j_i^{n+1/2} = F(\rho_i^{n+1/2}).$$

It is remarkable that, although we have an implicit scheme, we can avoid the solution of nonlinear algebraic equations. In fact, $\rho_i^{n+1/2}$ can be computed from the first equation (4), which is homogeneous, whereas for $j_i^{n+1/2}$ we have a second order equation that can be solved explicitly thanks to the knowledge of the unique asymptotic state.

On the contrary, we must take special care to choose a proper integration method for the source term in the corrector step in order to obtain a stable and second order scheme uniformly in ϵ. In fact, we must keep the source term implicit in the fluid dynamic regime to provide the correct asymptotic limit and stability, whereas in the rarefied regime the method must preserve the second order accuracy of the original Nessyahu-Tadmor schemes.

The general form of the corrector step can be written as

$$\begin{aligned} w_{i+\frac{1}{2}}^{n+1} &= \frac{1}{2}[w_i^n + w_{i+1}^n] + \frac{1}{8}[w'_i - w'_{i+1}] - \lambda[f(w_{i+1}^{n+\frac{1}{2}}) - f(w_i^{n+\frac{1}{2}})] \\ &+ \frac{1}{\epsilon \Delta x} \int_{x_i}^{x_{i+1}} \int_t^{t+\Delta t} s(w(x,t))\, dx\, dt. \end{aligned} \tag{20}$$

It has been shown [1, 14] that the simple (Crank-Nicholson) trapezoidal rule in time gives a scheme which is second order accurate in the rarefied regime but deteriorates to first order in the fluid limit. A similar result holds also for the midpoint integration rule in time applied to the source term. The stability property that is required for the integration of the source term in time is similar to the L-stability of numerical schemes for ordinary differential equations. In addition we must satisfy the second order accuracy requirements.

A choice that allow us to obtain the needed properties is to use a linear interpolation, often called *θ-method*, between the time levels n and $n+1$ in the evaluation of the stiff source term. Hence in the corrector step we approximate

$$\frac{1}{\epsilon \Delta x} \int_{x_i}^{x_{i+1}} \int_{t}^{t+\Delta t} s(w(x,t))\, dx\, dt \approx \frac{\Delta t}{2\epsilon} [\, 2\, \Theta\, s(w_{i+\frac{1}{2}}^{n+1}) + (I - \Theta)(s(w_{i+1}^{n}) + s(w_i^n))], \tag{21}$$

where, for the space integration, we have used the midpoint rule for the implicit term and the trapezoidal rule for the explicit term because of the staggered grid. Here $\Theta = \mathrm{diag}\{\theta_1,\, \theta_2\}$ is required to be a function of $\Delta t/\epsilon$ such that the scheme has the correct asymptotic behavior

$$\lim_{\Delta t/\epsilon \to +\infty} \theta_i = 1, \quad i = 1, 2, \tag{22}$$

and satisfies the second order accuracy condition in the rarefied regime

$$\theta_i \approx \frac{1}{2} \quad \text{for} \quad \frac{\Delta t}{\epsilon} \approx 1, \quad i = 1, 2. \tag{23}$$

In particular it can be shown [14] that if Θ satisfies (22) then in the limit $\epsilon \to 0$ the scheme we obtain corresponds to the standard Nessyahu-Tadmor scheme applied to the conservation law (11). This is clearly a high desirable property since it guarantees the consistency and the high order accuracy of the limiting scheme.

However, the choice of an optimal Θ is in general a difficult problem. One possibility is to to study the corresponding problem for a system of stiff ode's.

To this aim, let us consider the system of ordinary differential equations

$$\partial_t W = R(W), \tag{24}$$

supplemented with the initial condition $W(t=0) = W_0$.

In (24) W and R are N-dimensional vectors and we assume, without loss of generality, that $R(W_0) = 0$. If R is differentiable with respect to W, then the local behavior of (24) is described by the system $\partial_t W = JW$, where $J = \partial_W R|_{W=W_0}$ is the Jacobian matrix. If we assume J to be diagonalizable, with a change of variables we can consider the system

$$\partial_t V = MV, \tag{25}$$

where $M = \mathrm{diag}\{\mu_1, \ldots, \mu_N\}$ is defined by the eigenvalues of J.

Using the $\theta-$method to integrate (25) on the time interval $[t^n,\, t^{n+1}]$ we get

$$V^{n+1} = (I - \Theta \Delta t M)^{-1}(I + (I - \Theta)\Delta t M)V^n, \tag{26}$$

where now $\Theta = \text{diag}\{\theta_1, \ldots, \theta_N\}$.

Since the exact solution to system (25) on the same time interval is

$$V^{n+1} = e^{\Delta t M} V^n, \tag{27}$$

we derive as optimal values for θ_i the quantities

$$\theta_i = \frac{1}{\Delta t \mu_i} - \frac{1}{e^{\mu_i \Delta t} - 1}, \quad i = 1, \ldots, N. \tag{28}$$

It is easy to check that $0 \leq \theta_i \leq 1$ and that for $\mu_i \leq 0$, the definition (28) will satisfy properties (22-23).

The previous analysis for the system of two ode's corresponding to the space homogeneous equations (4-5) gives the values

$$\mu_1 = 0, \quad \mu_2 = -\frac{1}{\epsilon}(\alpha + \beta + \gamma j), \tag{29}$$

in particular $\mu_2 \leq 0$ in the TVD region of system (4-5) [4].

In summary, the family of predictor corrector central differencing schemes are defined by (19-20-21) with θ_i characterized by (28-29). Note that due to the absence of a source term in the first equation in practice only θ_2 is used.

We conclude this section by observing that in the TVD region of system (4-5) we have $|J| \leq (\alpha + \beta)/\gamma$ and hence $|\mu_2| \leq 2(\alpha + \beta)/\epsilon$. In all the numerical tests we have considered $\alpha + \beta = O(1)$ and uniform second order accuracy is obtained with the simplest choice of $\mu_2 = -1/\epsilon$, independently of the values of α, β, γ and j. This choice, although non optimal, has the advantage to avoid the computation of the eigenvalues associated to the relaxation operator. Obviously the validity of this approximation depends on the possibility to estimate a priori the eigenvalues of $s(w)$. However, it is interesting to remark that this can be done for a large class of kinetic equations [3].

4 Central schemes for the diffusive limit

In order to extend the schemes developed in the previous section to the diffusive case it is necessary to overcome the additional difficulty of the stiff convection term. A numerical study shows that in the diffusive case, the schemes constructed in previous section reduce to first order accuracy as $\epsilon \to 0$ [14]. This can be easily understood, since they compute the stiff spatial derivatives at time levels n in the predictor step and $n + 1/2$ in the corrector step and this will lead to the wrong asymptotic behavior. Similarly to the previous case we need an implicit evaluation also of the spatial derivatives to guarantee the correct equilibrium projection at a numerical level.

In order to overcome this problem it is necessary to modify the basic structure of the Nessyahu-Tadmor schemes. In fact, in the corrector step we need the knowledge of the numerical fluxes for the spatial derivatives at time $n+1$. This values are not provided by the schemes in the form (15-16).

To this aim we have developed a new class of central schemes in predictor-corrector form that essentially preserve all the properties of accuracy and stability of the Nessyahu-Tadmor schemes [14]. They are based on a predictor on the fluxes at time level $n+1$ and on a corrector that uses an implicit evaluation of the fluxes through the trapezoidal rule. For the scalar case (14) these schemes take the form

$$w_i^{n+1} = w_i^n - \lambda f_i', \tag{30}$$

$$w_{i+\frac{1}{2}}^{n+1} = \frac{1}{2}[w_i^n + w_{i+1}^n] + \frac{1}{8}[w_i' - w_{i+1}'] - \frac{\lambda}{2}[f(w_{i+1}^{n+1}) + f(w_{i+1}^n) - f(w_i^{n+1}) - f(w_i^n)]. \tag{31}$$

Then, we can extend the methodology used in the previous section to the case of a system like (4-5) under the scaling (9) using a fully implicit predictor and a linear interpolation both on the fluxes and on the source terms in the correction step.

This can be done by rewriting the system in compact form as

$$\partial_t w + \frac{1}{\epsilon^2}\partial_x(f_\epsilon(w)) = \frac{1}{\epsilon^2}s(w), \tag{32}$$

where w, f_ϵ and s are two dimensional vectors given by $w = (\rho, j)$, $f_\epsilon(w) = (\epsilon^2 j, \rho)$ and $s(w) = (0, \hat{J}_1)$.

We will omit here the details of the derivation of the schemes, since it follows the same lines of the previous section. The resulting predictor step reads

$$w_i^{n+1} = w_i^n - \frac{\lambda}{\epsilon^2}f_i' + \frac{\Delta t}{\epsilon^2}s(w_i^{n+1}), \tag{33}$$

with the numerical derivative f' is evaluated at time t^{n+1}, whereas the corrector step takes the form

$$\begin{aligned} w_{i+\frac{1}{2}}^{n+1} &= \frac{1}{2}[w_i^n + w_{i+1}^n] + \frac{1}{8}[w_i' - w_{i+1}'] \\ &-\frac{\lambda}{\epsilon^2}[\Phi(f(w_{i+1}^{n+1}) - f(w_i^{n+1})) + (I-\Phi)(f(w_{i+1}^n) - f(w_i^n))] \\ &+\frac{\Delta t}{2\epsilon^2}[\,2\,\Theta\, s(w_{i+\frac{1}{2}}^{n+1}) + (I-\Theta)(s(w_{i+1}^n) + s(w_i^n))], \end{aligned} \tag{34}$$

where $\Phi = \mathrm{diag}\{\phi_1, \phi_2\}$.

We point out that in order to get a scheme explicitly implementable we have to deal with the fully implicit predictor step on both the fluxes and the source term. If the numerical derivative is nonlinear this will require the solution of nonlinear algebraic equations. To avoid this we have used a modified predictor step based on the semi-implicit discretization developed in [11]. In this scheme the numerical derivative for j is taken at time level n whereas the numerical derivative for ρ is taken at time $n+1$. This will preserve the correct asymptotic behavior of the predictor step since in the limit gives

$$j_i^{n+1} = F\left(\rho_i^{n+1}, \frac{1}{\Delta x}\rho_i'(t^{n+1})\right). \tag{35}$$

Finally, we emphasize that the choice of the optimal integration parameters for the fluxes cannot be done exactly as in the case of a stiff source term. In fact, the choice of the optimal Φ leads to the analogous of equation (28) with μ_i replaced by ϕ_i but the computation of the eigenvalues will now depend on the spatial derivatives.

Since these eigenvalues are proportional to the modulus of the characteristic speeds $1/\epsilon$ of the original system it will be natural to use as optimal parameters $\phi_1 = \phi_2 = -1/\epsilon$. In the limit as $\epsilon \to 0$ this choice will lead to a fully implicit corrector step for the limiting nonlinear parabolic equations. Note that this choice will reduce the accuracy of the limiting schemes to first order in time. This may be a drawback when the diffusion term is small and the usual CFL parabolic condition $\Delta t \approx (\Delta x)^2$ can be avoided.

However, other choices are possible. For example, the choice $\phi_1 = 0$ and $\phi_2 = -1/\epsilon^2$ will also give a scheme which is second order uniformly accurate and with the correct asymptotic limit [14]. In particular when the diffusion term vanishes this last choice guarantees that the limiting equilibrium scheme corresponds to the modified Nessyahu-Tadmor scheme applied to the resulting conservation law.

5 Numerical results

In this section we present numerical results that demonstrate the performance of the schemes for the kinetic equations presented in sections 2. All the numerical examples presented here refer to the the optimal choices of integration parameters we have derived in the previous sections and to the schemes provided with the non-oscillatory numerical UNO limiter [5]

$$\rho_i' = \mathrm{MM}\{\Delta\rho_{i-\frac{1}{2}} + \frac{1}{2}\mathrm{MM}\{\Delta^2\rho_{i-1}, \Delta^2\rho_i\}, \Delta\rho_{i+\frac{1}{2}} - \frac{1}{2}\mathrm{MM}\{\Delta^2\rho_{i+1}, \Delta^2\rho_i\}\}, \tag{36}$$

where MM denotes the Min-Mod function and $\Delta\rho_i = \rho_{i+\frac{1}{2}} - \rho_{i-\frac{1}{2}}$. As usual, the reference 'exact' solutions have been obtained with much finer grids.

5.1 Fluid dynamic case

Consider system (4-5) in the fluid dynamic scaling with initial data

$$\begin{aligned} &\rho_L = 1.0, \quad j_L = 0, \quad -1 < x < 0, \\ &\rho_R = 0, \quad j_R = 0, \quad 0 < x < 1. \end{aligned} \tag{37}$$

The numerical computations refers to $t = 0.5$ with $\Delta x = 0.01$ and $\lambda = 0.5$. The solutions for the mass ρ and the flux j in the rarefied regime ($\epsilon = 1$), intermediate regime ($\epsilon = 0.1$) and fluid dynamic regime ($\epsilon = 10^{-8}$) are depicted in figure 1 and 2. In the limit we obtain a Riemann problem for different nonlinear hyperbolic equations depending on the choice of the parameters α, β, and γ. Figure 1 refers to the choice $\alpha = 0.5$, $\beta = 1.5$, $\gamma = 0$ which in the limit correspond to a simple linear advection equation whereas figure 2 refers to $\alpha = 0.5$, $\beta = 1.5$, $\gamma = 2$ which in the limit gives a a nonlinear conservation law. Clearly in both cases the schemes describes with high accuracy the solution of the system also in the under-resolved regime.

5.2 Diffusive case

Next we have considered system (4-5) in the diffusive scaling.

If we choose $\alpha_\epsilon = 1 - \epsilon$, $\beta_\epsilon = 1 + \epsilon$ and $\gamma_\epsilon = 0$ in the limit $\epsilon \to 0$ we obtain a linear advection-diffusion equation. The solutions are computed at time $t = 3.0$ for the initial data

$$\begin{aligned} &\rho_L = 4.0, \quad j_L = 0, \quad -10 < x < 0, \\ &\rho_R = 2.0, \quad j_R = 0, \quad 0 < x < 10. \end{aligned} \tag{38}$$

In figure 3 we report the numerical results for $\Delta x = 0.1$ and $\lambda = 0.25$ in the intermediate regime ($\epsilon = 0.5$), and $\lambda = 0.001$ in the diffusive regime ($\epsilon = 10^{-6}$).

Finally taking $\alpha_\epsilon = \beta_\epsilon = 1$, $\gamma_e = 2\epsilon$, in the limit as $\epsilon \to 0$ we obtain the viscous Burgers equation. The initial conditions is given by two local equilibrium states characterized by

$$\begin{aligned} &\rho_L = 2.0, \quad -10 < x < 0, \\ &\rho_R = 1.0, \quad 0 < x < 10, \end{aligned} \tag{39}$$

with $j = [(1 + \rho^2\epsilon^2)^{\frac{1}{2}} - 1] / \epsilon^2$.

Figure 4 shows the numerical solutions at time $t = 2.0$ for $\Delta x = 0.1$ and $\lambda = 0.2$ in the intermediate regime ($\epsilon = 0.4$), $\lambda = 0.001$ in the diffusive regime ($\epsilon = 10^{-6}$).

Again the schemes capture very well the dynamic of the solution in all the different regimes.

6 Conclusions

We have introduced a way of constructing under-resolved numerical schemes for several kinetic equations with multiple scales. On the contrary to previous approaches, this new class of schemes uses a predictor corrector technique to obtain the uniform second order accuracy in space and time with respect to the scaling parameters. In the fluid-dynamic case they can be constructed using directly the central schemes recently developed by Nessyahu and Tadmor [13] whereas in the diffusive case a modified version of the central schemes has to be used.

The implicit treatment of the stiff terms is done using a linear interpolation which depends on the scaling parameter. This gives a class of explicitly implementable schemes which are stable independently on the small relaxation parameter and possess the correct asymptotic limit in both the fluid-dynamic and diffusive scaling. In the future we will extend this idea to more general kinetic equations and multidimensional problems.

References

[1] R.E. Caflish, S. Jin and G. Russo, *Uniformly accurate schemes for hyperbolic systems with relaxation*, SIAM J. Num. Anal., 34, 1, pp. 246-281, (1997).

[2] C. Cercignani, Theory and application of the Boltzmann equation, *Springer-Verlag, New York* (1988)

[3] E. Gabetta, L. Pareschi and G. Toscani, *Relaxation schemes for nonlinear kinetic equations*, SIAM J. Num. Anal., 34, 6, pp. 2168-2194, (1997).

[4] E. Gabetta and B. Perthame, *Scaling limits of the Ruijgrok-Wu model of the Boltzmann equation*, Proceedings of the International Conference on Nonlinear Equations and Applications, Bangalore, 19-23 August 1996, Springer-Verlag.

[5] A. Harten and S. Osher, *Uniformly high order accurate non-oscillatory schemes. I*, SIAM J. Num. Anal., 24, pp. 229-309, (1982).

[6] S. Jin, *Runge-Kutta methods for hyperbolic conservation laws with stiff relaxation terms*, J. Comp. Phys., 122, pp.51-67, (1995).

[7] S. Jin, L. Pareschi and G. Toscani, *Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations*, SIAM J. Num. Anal., 35, 6, pp. 2405-2439, (1998).

[8] S. Jin, L.Pareschi and G. Toscani, *Uniformly accurate diffusive relaxation schemes for multiscale transport equations*, SIAM J. Num. Anal. submitted.

[9] A. Klar, *An asymptotic-induced scheme for nonstationary transport equations in the diffusive limit*, SIAM J. Num. Anal., 15, pp. 1073-1094, (1998).

[10] P.L. Lions and G. Toscani, *Diffusive limit for finite velocities Boltzmann kinetic models*, Rev. Math. Iberoamer., 13, pp. 473-513, (1997).

[11] G. Naldi and L. Pareschi, *Numerical schemes for kinetic equations in diffusive regimes*, Appl. Math. Lett., 11, pp. 29-35, (1998).

[12] G. Naldi and L. Pareschi, *Numerical schemes for hyperbolic systems with stiff diffusive relaxation*, Pubblicazioni IAN, 1051, (1997).

[13] H. Nessyahu and E. Tadmor, *Non-oscillatory central differencing for hyperbolic conservation laws*, J. Comp. Phys., 87, pp. 408-463 (1990).

[14] L. Pareschi and M. Ronconi, *Predictor-corrector central schemes for multiscale hyperbolic and kinetic equations*, work in progress.

[15] W. Ruijgrok and T.T. Wu, *A completely solvable model of the nonlinear Boltzmann equation*, Phys A, 113, pp. 401-416, (1982).

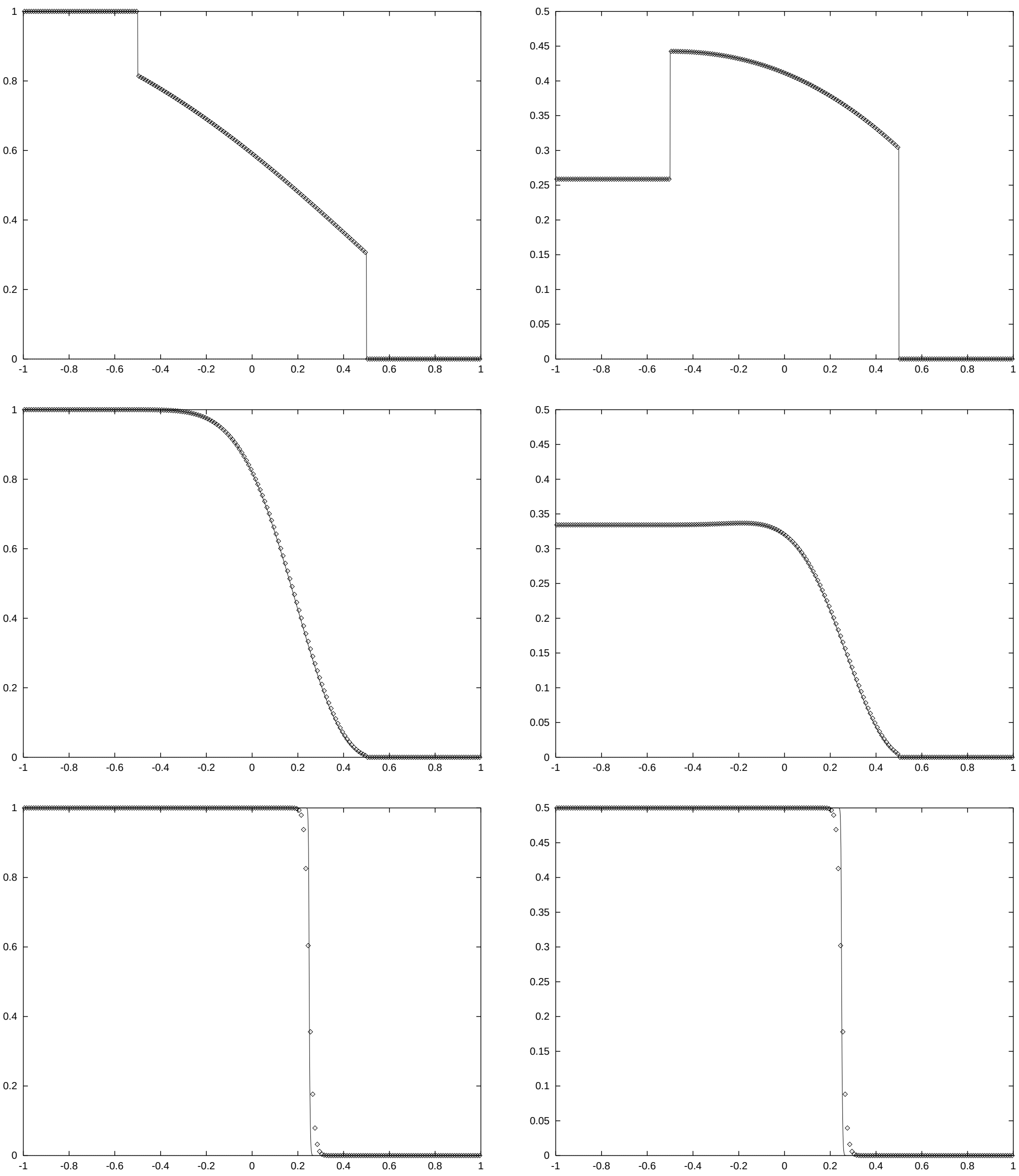

Figure 1: Fluid dynamic scaling : numerical solutions for the mass ρ (left) and the flux j (right) with $\alpha = 0.5$, $\beta = 1.5$ and $\gamma = 0$. From top to bottom : rarefied regime ($\epsilon = 1$), intermediate regime ($\epsilon = 0.1$), and fluid-dynamic regime ($\epsilon = 10^{-8}$). Initial data (37).

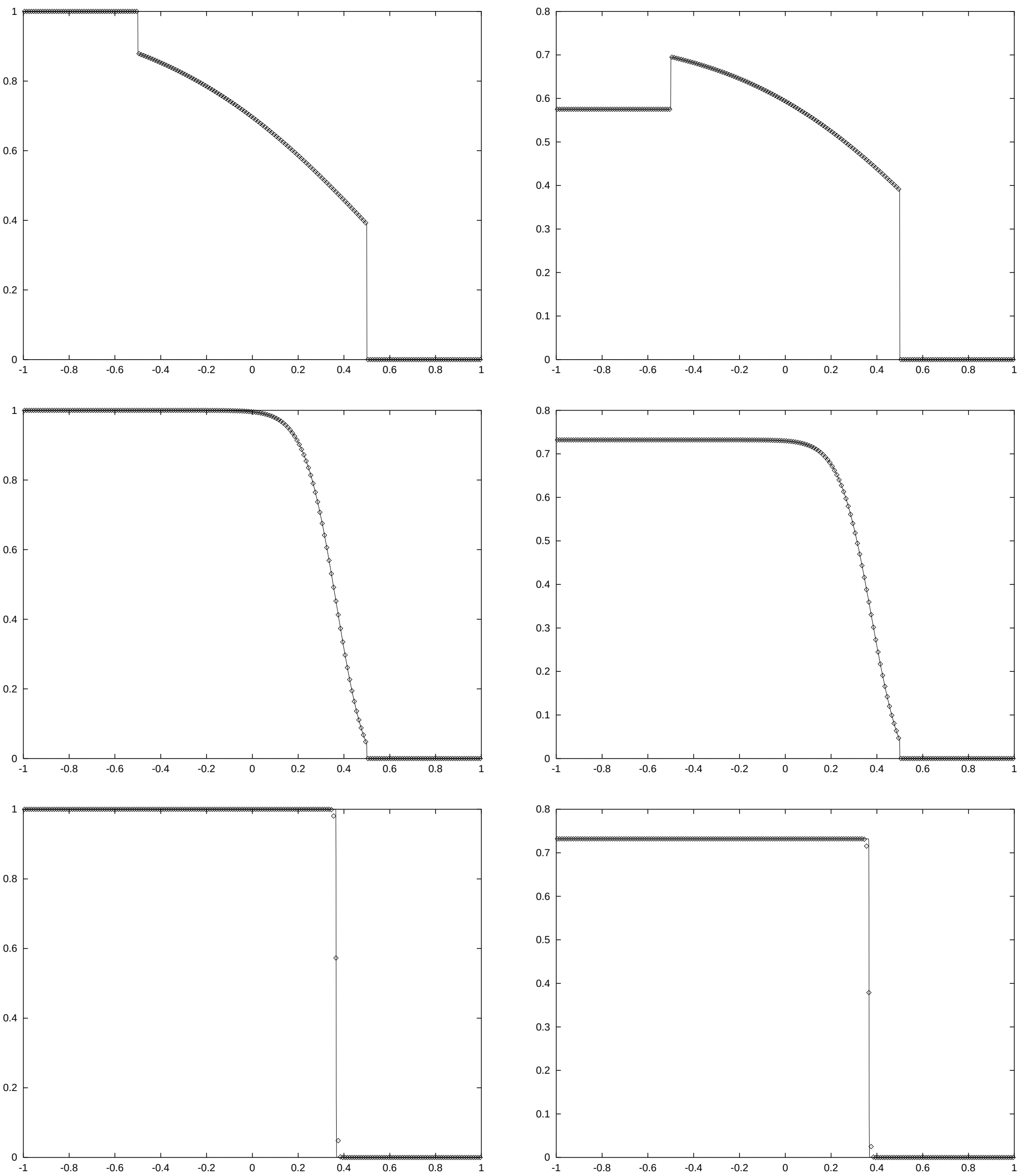

Figure 2: Fluid dynamic scaling : numerical solutions for the mass ρ (left) and the flux j (right) with $\alpha = 0.5$, $\beta = 1.5$ and $\gamma = 2$. From top to bottom : rarefied regime ($\epsilon = 1$), intermediate regime ($\epsilon = 0.1$) and fluid-dynamic regime ($\epsilon = 10^{-8}$). Initial data (37).

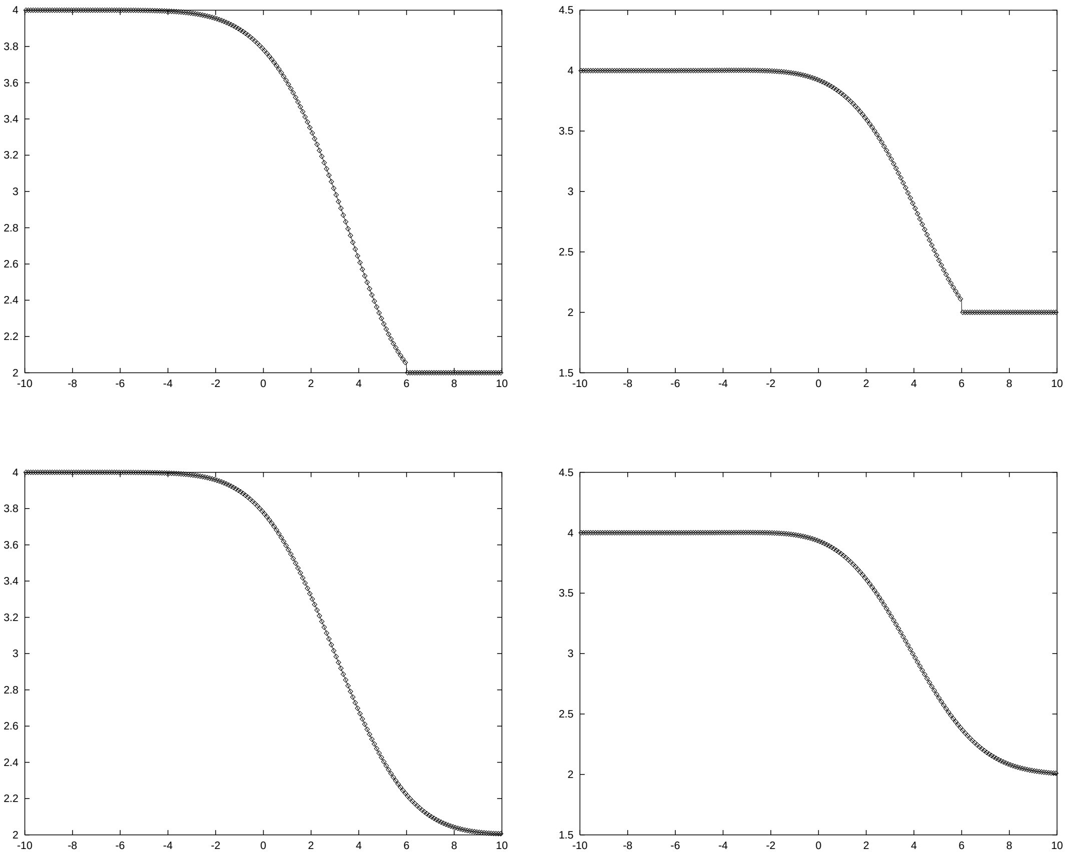

Figure 3: Diffusive scaling : numerical solutions for the mass ρ (left) and the flux j (right) with $\alpha_\epsilon = 1-\epsilon$, $\beta_\epsilon = 1+\epsilon$ and $\gamma_\epsilon = 0$. From top to bottom : intermediate regime ($\epsilon = 0.5$) and diffusive regime ($\epsilon = 10^{-6}$). Initial data (38).

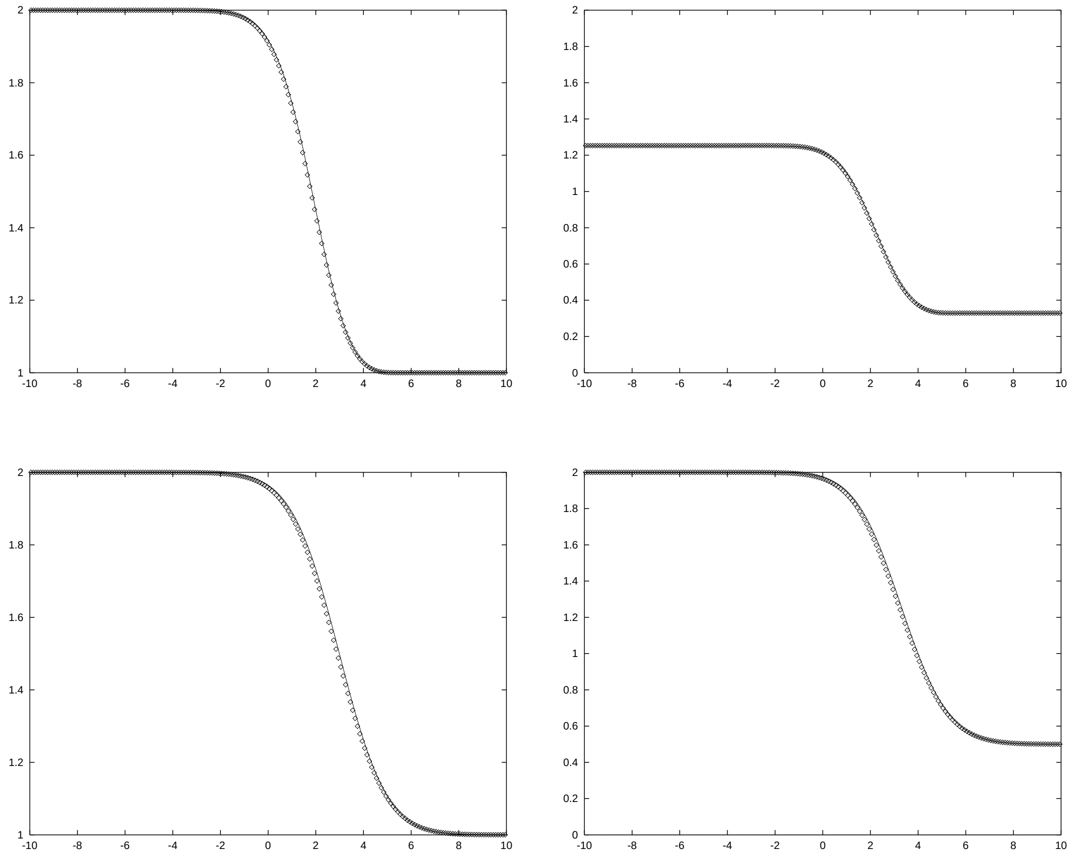

Figure 4: Diffusive scaling : numerical solutions for the mass ρ (left) and the flux j (right) with $\alpha_\epsilon = 1$, $\beta_\epsilon = 1$ and $\gamma_\epsilon = 2\epsilon$. From top to bottom : intermediate regime ($\epsilon = 0.4$) and diffusive regime ($\epsilon = 10^{-6}$). Initial data (39).